Glauber dynamics of FK percolation and a new bound for the critical point when $q < 1$

Corentin Faipeur and Tejas Oke
joint work with Vincent Beffara

Saint-Flour summer school, 2026

FK percolation model

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- $(\Lambda, E(\Lambda))$ a finite graph, $p \in [0,1]$, $q > 0$.
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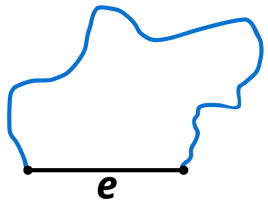
$$\phi^{\xi}_{\Lambda,p,q}(\omega) \propto q^{k(\omega,\xi)} \prod_{e \in E(\Lambda)} p^{\omega_e}(1-p)^{1-\omega_e}$$

  where $k(\omega, \xi) =$ number of **clusters** in $\omega \cup \xi$.
- We add a boundary condition ξ.

Conditional law

- For every edge $e \in E(\Lambda)$,

$$\phi^{\xi}_{\Lambda,p,q}(\omega_e = 1 \mid \omega|_{E(\Lambda)\setminus\{e\}}) = \begin{cases} p & \text{if } \\ \dfrac{p}{p+(1-p)q} =: p' & \text{otherwise} \end{cases}$$



NB: $q < 1 \iff p < p'$.

Conditional law and comparison inequalities

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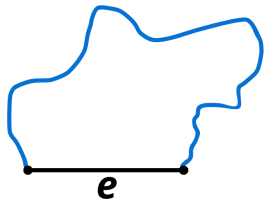

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- Stochastic dominations:

$$\bigotimes_{e\in E(\Lambda)} \mathcal{B}(\min(p,p')) \preceq \phi^{\xi}_{\Lambda,p,q} \preceq \bigotimes_{e\in E(\Lambda)} \mathcal{B}(\max(p,p'))$$

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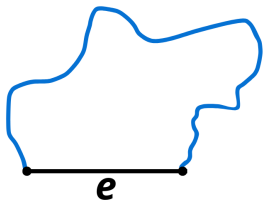

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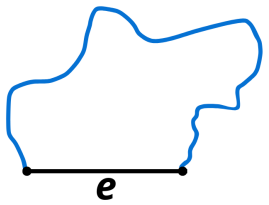

We denote by $\mathrm{FK}(p,q)$ the set of measures on $\{0,1\}^{E(\mathbf{Z}^d)}$ obtained by either of these methods.

Two questions

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2. For which p and q is there an infinite cluster?

Phase transition for $q \geq 1$

For $q \geq 1$, we know how to answer thanks to the FKG inequalities.

There exists $p_c(q) \in (0,1)$ such that, for every $\phi_{p,q} \in \mathrm{FK}(p,q)$

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- non-uniqueness at $p = p_c(q)$ for $q > 4$ [DGHMT21]

And for $q < 1$?

For $q < 1$, FKG is no longer valid!

For every measure $\phi_{p,q} \in \mathrm{FK}(p, q)$,

$$\bigotimes_{e \in E(\mathbf{Z}^d)} \mathcal{B}(p) \preceq \phi_{p,q} \preceq \bigotimes_{e \in E(\mathbf{Z}^d)} \mathcal{B}(p')$$

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Improved comparison inequality

Theorem ([BFO25])

On $\mathbf{Z}^d$, $d \geq 2$, for every $p \in (0,1)$ and $q < 1$, there exist $\widetilde{E}$ and $\widehat{E}$ two subsets of edges of positive density and $\widehat{\varepsilon}, \widetilde{\varepsilon} > 0$ such that

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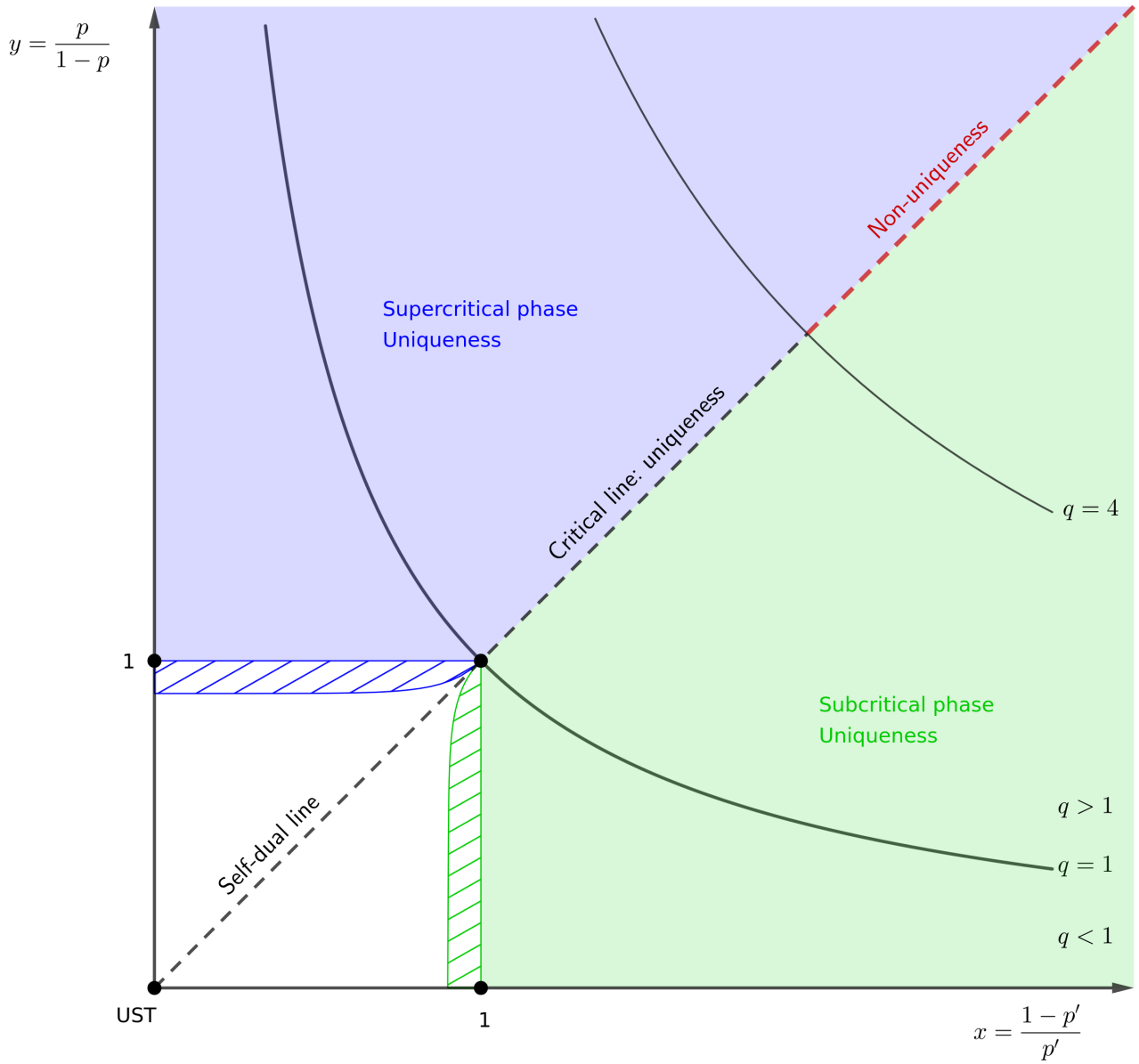
Thanks to the theory of "essential enhancements" [AG91].

Phase diagram in 2D

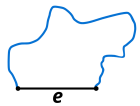


Glauber dynamics for FK percolation

- $X_0 = \omega \in \{0,1\}^{E(\Lambda)}$
- At time t, update of edge e:

$$X_t(e') = X_{t-}(e') \ \forall e' \neq e \text{ and}$$

$$X_t(e) = \begin{cases} 1 & \text{if } U_t \leq p \text{ and } \\ 1 & \text{if } U_t \leq p' \text{ and } \\ 0 & \text{otherwise} \end{cases}$$



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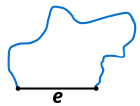

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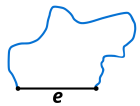

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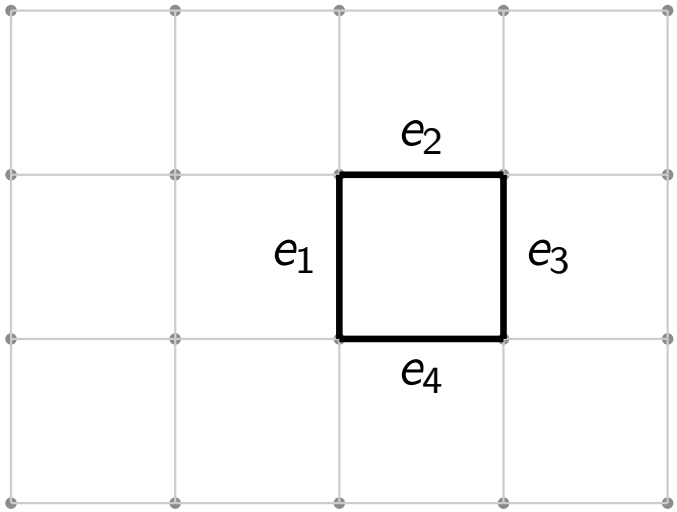
- Invariant law: $\bigotimes_{e \in E(\Lambda)} \mathcal{B}(p')$

For every t we have $X_t \leq Y_t$, hence we recover $\phi^{\xi}_{\Lambda,p,q} \preceq \bigotimes_{e \in E(\Lambda)} \mathcal{B}(p')$.

A better coupling with a product measure



At time t, update a device of 4 edges

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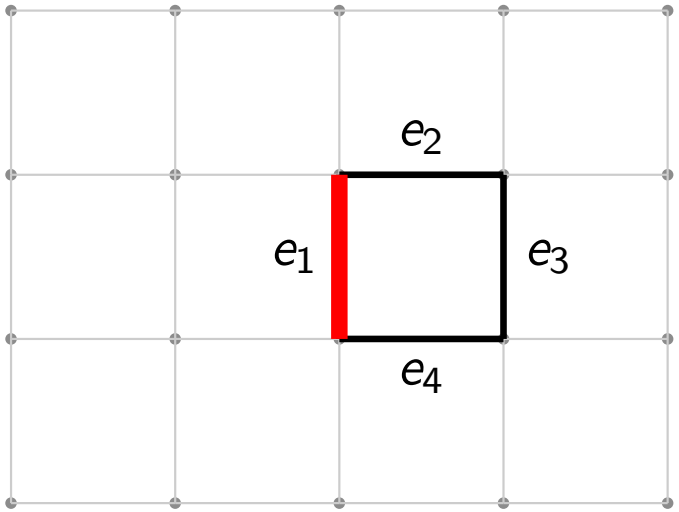

Resample e_1 with U_1

Standard updates of e_1, e_2, e_3 in X_t and Y_t.

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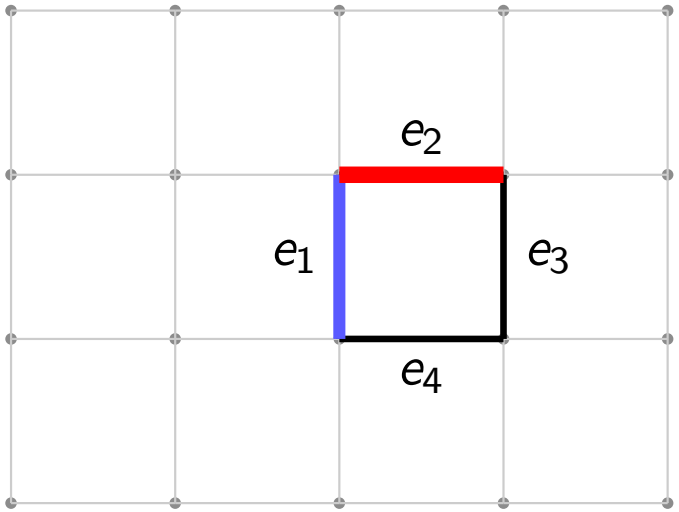

Resample e_2 with U_2

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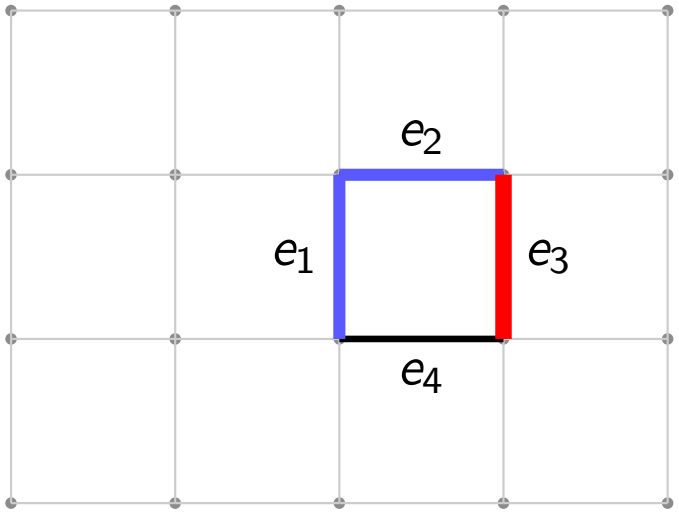

Resample e_3 with U_3

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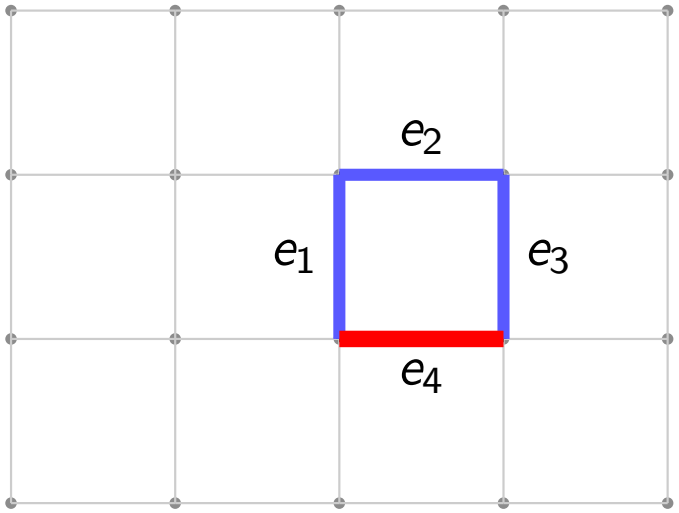

Resample e_4 with U_4

If $U_1, U_2, U_3 \leq p$, then the endpoints of e_4 are connected in X_t off e_4, so we can set

$$Y_t(e_4) = \mathbf{1}_{\{U_4 \leq p\}},$$

while preserving the ordering. Otherwise use the standard p' update.

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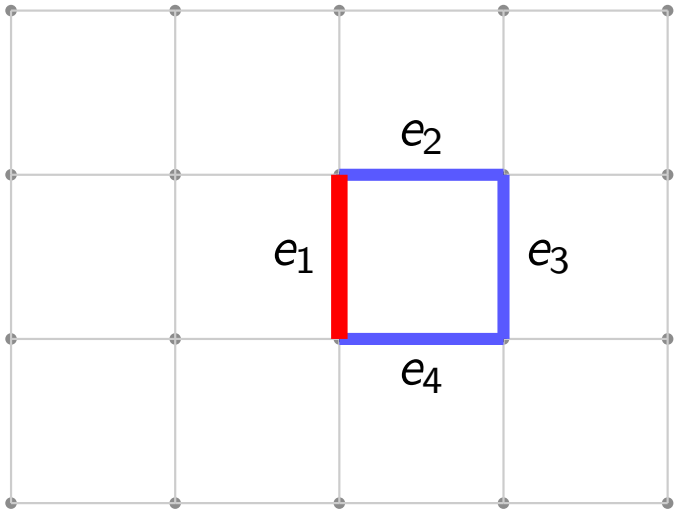

Resample e_1 with U_5

Refresh e_1, e_2, e_3 in Y_t to recover independence of the $Y_t(e_i)$ and in X_t to preserve ordering.

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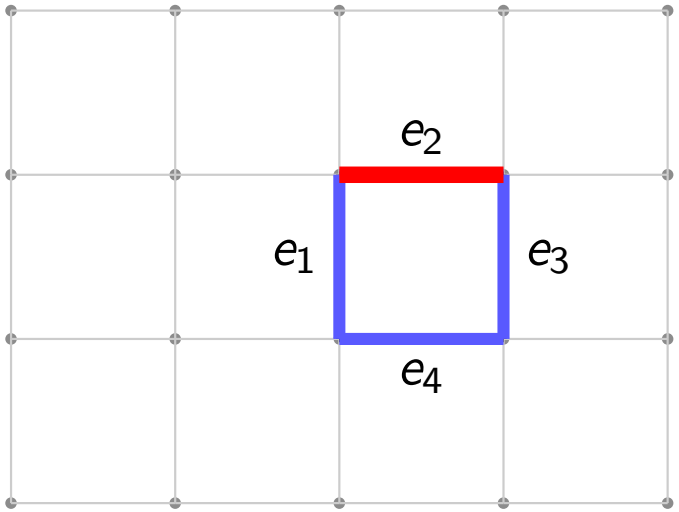

Resample e_2 with U_6

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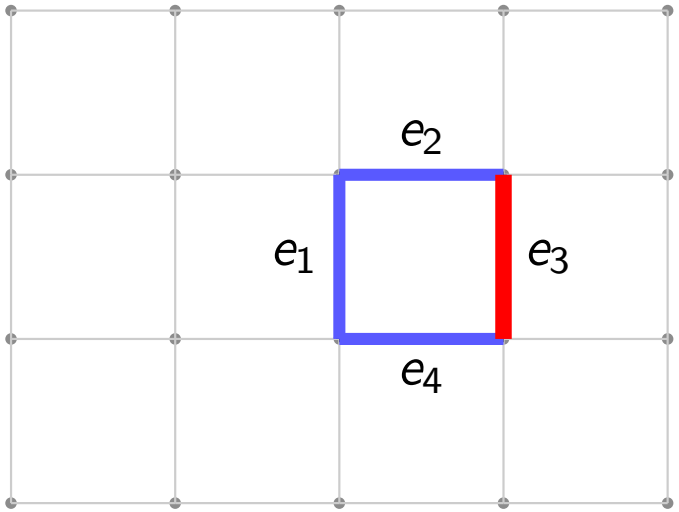

Resample e_3 with U_7

Refresh e_1, e_2, e_3 in Y_t to recover independence of the $Y_t(e_i)$ and in X_t to preserve ordering.

Improved comparison inequality $q < 1$

Invariant law of (X_t): still $\phi^{\xi}_{\Lambda,p,q}$.
Invariant law of (Y_t):

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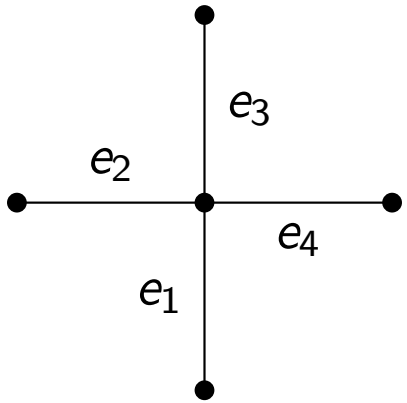
For the domination from below, the same thing with the device



Thank you for your attention!!

[AG91] Michael Aizenman and Geoffrey Grimmett. "Strict monotonicity for critical points in percolation and ferromagnetic models". In: *J. Statist. Phys.* 63.5-6 (1991), pp. 817–835.

[BD12] Vincent Beffara and Hugo Duminil-Copin. "The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$". In: *Probab. Theory Related Fields* 153.3-4 (2012), pp. 511–542.

[BFO25] Vincent Beffara, Corentin Faipeur, and Tejas Oke. *A new bound for the critical point of the FK model for $q < 1$*. 2025. arXiv: 2512.16486 [math.PR].

[DGHMT21] Hugo Duminil-Copin, Maxime Gagnebin, Matan Harel, Ioan Manolescu, and Vincent Tassion. "Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$". In: *Ann. Sci. Éc. Norm. Supér. (4)* 54.6 (2021), pp. 1363–1413.

[DST17] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion. "Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$". In: *Comm. Math. Phys.* 349.1 (2017), pp. 47–107.